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System Normalizers and Carter Subgroups

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1. INTRODUCTION

Two important conjugacy classes of subgroups of solvable groups are the system normalizers introduced by P. Hall [*5*] and the nilpotent self-normalizing subgroups discovered by R. Carter [*3*]. The latter type of subgroups are called Carter subgroups. Both these conjugacy classes of subgroups may be thought of as analogs of the Cartan subalgebras of Lie algebras and for this reason and others the connection between these two classes has been studied. In this direction, Carter [*3*] has shown that each system normalizer is contained in a Carter subgroup and that each Carter subgroup contains a system normalizer. Furthermore, he was able to give strong information about this containment relation for special classes of groups. He showed [*4*] that in solvable A-groups each Carter subgroup contains a unique system normalizer and that in solvable groups of nilpotent length at most three there is a unique Carter subgroup containing a given system normalizer. Huppert [*6*] has also given a special method of constructing certain system normalizers contained in a Carter subgroup. Specifically, he showed that if C is a Carter Subgroup of a solvable group and $\mathscr{S}$ is a Sylow system reducible into C then $D = N(\mathscr{S})$, the normalizer of $\mathscr{S}$, is contained in C. These results stimulate two questions. First, does every system normalizer D contained in a Carter subgroup C arise by the construction given by Huppert? Second, for a general solvable group how many system normalizers are contained in a single Carter subgroup and how many Carter subgroups contain a system normalizer? These questions and several other problems we shall answer by proving some general theorems about solvable groups.

In this paper all groups mentioned are implicitly assumed to be finite. The notation and definitions are standard (see [*1*, *4*]).

In order to carry out the program outlined above certain new concepts must

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be introduced. Let G be a solvable group and let p be a prime dividing the order $|G|$ of G. Let E be a p-subgroup of G which normalizes some p-complement S of G. Suppose that H is a p'-subgroup of G also normalized by E. We shall say that H is *extendible* if there is a p-complement T of G which contains H and is also normalized by E. The study of conditions on G, E or H which guarantee the extendibility of H will be shown to be quite relevant to the problems mentioned above as well as other problems in the theory of solvable groups.

With this definition and the same notation the three main theorems of this paper are as follows:

THEOREM A. *If G has p-length one then H is extendible.*

THEOREM B. *If E centralizes H then H is extendible.*

THEOREM C. *There exist solvable groups G with subgroups E and H as above such that H is not extendible.*

The first two theorems show that restrictions on the structure of G or on the action of E on H will guarantee the extendibility of H. The third result proves that such hypotheses are necessary. However, these theorems raise some questions which we cannot answer. Let E be a p-subgroup of a solvable group G and suppose that E normalzes some p-complement of G. The most general problem is the determination of the p'-subgroups of G which are maximal subject to being normalized by E. The first theorem states that these are all p-complements of G if G has p-length one. The general situation is not clear to us at all.

We shall now describe the theorems which we are able to derive from the above results. In a previous paper [*1*] we showed that there is a solvable group G with a subgroup H and a system normalizer D contained in H such that D normalizes *no* system of H. Our example was a group of p-length one for all primes p. However, Theorem A enables us to prove for groups of p-length one for all p that any system of H which D should normalize arises in a natural way.

THEOREM 1. *Let G be a solvable group of p-length one for all primes p. Let D be a system normalizer of G and H be a subgroup of G. If D is contained in H and normalizes the system $\mathscr{T}$ of H then there is a system $\mathscr{S}$ of G normalized by D with $\mathscr{S}$ reducing into $\mathscr{T}$.*

We have been unable to remove the hypothesis on p-length and we strongly suspect that the result is false without such a restriction. However, there may always be such an $\mathscr{S}$ reducing into some system of H, not necessarily $\mathscr{T}$.

We now turn to the main applications, a study of the containment relations between system normalizers and Carter subgroups. These results are consequences of Theorem B.

THEOREM 2. *If D and D_1 are system normalizers of the solvable group G and D and D_1 are subgroups of the same Carter subgroup C of G then D and D_1 are conjugate subgroups of C.*

Since D and D_1 are system normalizers they are conjugate in G; we are asserting that they are even conjugate by an element of C. At this point we pause to point out the relevance of an example in a previous paper [*1*] to the above theorem. We there exhibited a solvable group G with system normalizer D and a homomorphic image of G in which the image of $N(D)$ was properly contained in the normalizer of the image of D. Were it not for this behavior, the above theorem and the consequences of it which we shall give below could be proved without the use of the theorems on extendibility. This concept was introduced to avoid the difficulties just mentioned. For example, if we attempt to prove Theorem 2 by induction then we may assume that G has a minimal normal subgroup M such that $DM = D_1M$. If $x \in G$ such that $D^x = D_1$ then $x \in N(DM)$. If the normalizer of D were homomorphism invariant then $N(D)M = N(DM)$ and we could assume that $x \in M$. In this case it is possible to show that one can assume $x \in C$, a Carter subgroup given to contain D and D_1, thus accomplishing a "proof" of Theorem 2.

The next result answers the question raised in the first paragraph of this paper.

THEOREM 3. *Let D be a system normalizer and C be a Carter subgroup of the solvable group G. If D is a subgroup of C then*

1. *the number of Carter subgroups containing D is equal to the index $|N(D) : N_C(D)|$,*

2. *the number of system normalizers contained in C is equal to the index $|C : N_C(D)|$ and*

3. *the number of systems normalized by D and reducing into C is equal to the index $|N_C(D) : D|$.*

In particular, notice that the third part not only shows that D arises by Huppert's construction but gives the exact number of ways this can happen. In the case of solvable A-groups we now have Carter's result:

THEOREM 4. *If G is a solvable A-group then each Carter subgroup of G contains a unique system normalizer.*

In fact, such a Carter subgroup will be a nilpotent A-group and so abelian. Therefore, each of its subgroups is normal so Theorem 2 gives this result immediately. For solvable groups of nilpotent length at most three we shall also give a short proof of Carter's theorem:

THEOREM 5. *Let G be a solvable group of nilpotent length at most three with a Carter subgroup C and a system normalizer D contained in C. If D is a subnormal subgroup of the subgroup H of G then H is contained in C.*

Since all subgroups of nilpotent groups are subnormal this result implies that C is the only Carter subgroup containing D. The next theorem shows that each system has attached to it in a natural way a unique Carter subgroup.

THEOREM 6. *If $\mathscr{S}$ is a system of the solvable group G then there is a unique Carter subgroup into which $\mathscr{S}$ is reducible.*

A final theorem in this direction is

THEOREM 7. *If D is the normalizer of the system $\mathscr{S}$ then reduces into $N(D)$.*

In another direction we shall give two results pertaining to the covering and avoidance properties of system normalizers.

THEOREM 8. *Let $p_1, p_2, \cdots, p_n$ be the distinct primes dividing the order $|G|$ of the solvable group G. For $i = 1, 2, \cdots, n$ let D_i be a Sylow p_i-subgroup of a system normalizer. If $D_1D_2 \cdots D_n$ is a group then it is a system normalizer.*

This theorem states that a subgroup which is "locally" a system normalizer is in fact such a subgroup. If this were not the case such products $D_1D_2 \cdots D_n$ would be examples of subgroups with the covering and avoiding properties which are not system normalizers. Theorem 8 is in fact a consequence of the more general

THEOREM 9. *Let H be a subgroup of a solvable group and suppose that H has the avoidance property. If H is not a subgroup of a system normalizer then for some prime p the Sylow p-subgroups of H normalize no p-complement of G.*

Our last theorem relates the Carter subgroups of a group and the Carter subgroups of a subgroup in certain special circumstances.

THEOREM 10. *Let M be a nonnormal maximal subgroup of a solvable group G. If G has nilpotent length at most three then there is a Carter subgroup C of G such that $C \cap M$ is a Carter subgroup of M.*

The symmetric group on three letters is an example which shows that the hypothesis on nonnormality cannot be dropped. Furthermore, the restriction on nilpotent length cannot be removed. Carter [*4*] gives an example of a solvable group of order $2^3 \cdot 3 \cdot 5^4$ and nilpotent length four the Carter subgroups of which have order $2^3 \cdot 5$. This group possesses a unique conjugacy class of non-normal maximal subgroups of order $2 \cdot 3 \cdot 5^4$ each of which has Carter subgroups of order $2 \cdot 5^2$. Since one hundred does not divide forty this example shows the need for the hypothesis on nilpotent length.

The organization of the remainder of this paper is as follows. The next section contains proofs of Theorems A and B while the third part is devoted to proving Theorems 1 through 10. A fourth section contains the example required for a demonstration of Theorem C and the paper will be concluded with a few remarks and suggestions for further work.

2. Proof of the Extension Theorems

In order to prove Theorem A we proceed by induction on the group order. The notation is as described: E is a p-subgroup of the solvable group G and E normalizes the p'-subgroup H and the p-complement S. Suppose that G has normal p'-subgroups. Let M be a minimal normal subgroup and a p'-subgroup. Thus EM/M normalizes the p-complement S/M of G/M so there is a p-complement T/M of G/M which contains HM/M and is normalized by EM/M. Hence, T is the desired p-complement of G. However, if G has no normal p'-subgroups then a Sylow p-subgroup P of G is normal as G has p-length one. Thus E is contained in P so E must centralize H as well as S. A p-complement of the centralizer $C(E)$ of E will be a p-complement of G. Any p-complement T of $C(E)$ containing H will suffice for the proof.

For the proof of Theorem B we use the same notation and assume E centralizes H. Therefore, we need only show that $S \cap C(E)$ is a p-complement of $C(E)$. For in that case, H is a subgroup of a p-complement K of $C(E)$ so $(S \cap C(E))^x = K$ for $x \in C(E)$ and S^x is a p-complement of G containing H and normalized by E since x centralizes E.

Thus we require only the

Lemma. *Let E be a p-subgroup of a solvable group G and suppose E normalizes a p-complement S of G. Then $S \cap C(E)$ is a p-complement of $C(E)$.*

Proof: Let M be a minimal normal subgroup of G and suppose that M is a p'-subgroup. Let C_1/M be the centralizer of EM/M in G/M. Now S contains M so by induction $S \cap C_1$ is a p-complement of C_1. Therefore, we need only show that a p-complement L of C_1, which is normalized by E, contains a p-complement of $C(E)$. Let K be a p-complement of $C(E)$. Now $C(E)$is contained in C_1 so

$$|K| \leq |K \cap M| \, |L : M| = |C(E) \cap M| \, |L : M|.$$

Therefore, if we can show that

$$|C(E) \cap L| = |C(E) \cap M| \, |L : M|$$

we will be done in this case. However, to do this it suffices to show that

$L = M(C(E) \cap L)$. For $(C(E) \cap L) \cap M = C(E) \cap M$ since L contains M so then

$$|C(E) \cap L| = |C(E) \cap L : C(E) \cap M|\,|C(E) \cap M| = |L : M|\,|C(E) \cap M|.$$

However, the p-group E normalizes the p'-group L and centralizes L/M. Therefore we need only show that $C_L(E)$ covers L/M. But this is well known to occur (e.g., see [*1*]).

We now suppose that M is a p-subgroup. Again let C_1/M be the centralizer of EM/M in G/M so by an application of the induction hypothesis $C_1 \cap S$ is a p-complement of C_1. Therefore, it suffices to show that E centralizes $C_1 \cap S$. However, the commutator group $(E, C_1 \cap S)$ is contained in M since E centralizes C_1/M and is contained in S since E normalizes S. It is therefore the identity subgroup and the lemma and Theorem B are proven.

3. Proofs of Theorems 1–10

In the proof of each theorem, the notation will be the same as that described in the statement of the result. The theorems will be proved in the same order as they were presented above, with the exception of Theorem 4 which was already proved.

In order to prove Theorem 1 we let p be a prime and T^p the p-complement of $\mathscr{T}$. It suffices to show that there is a p-complement of G containing T^p and normalized by D. However, the unique p-complement D^p of D normalizes T^p so D^p is a subgroup of T^p since T^p is a p-complement of H. Therefore, if D_p is the Sylow p-subgroup of D, we need only prove that there is a p-complement of G containing T^p and normalized by D_p. That this is true is an immediate consequence of Theorem A.

The proof of the second theorem requires a longer development. We first demonstrate the

LEMMA. *If H is an abnormal subgroup of the solvable group G then any two systems of G reducing into H are conjugate by an element of H.*

Proof. Let $H = H_0 < H_1 < \cdots < H_n = G$ be a maximal chain joining H and G. The abnormality of H guarantees that each H_i is not a normal subgroup of H_{i+1}. Consequently, if $\mathscr{T}_{i+1}$ is a system of H_{i+1} reducing into H_i then $D_{i+1} = N(\mathscr{T}_{i+1})$ is contained in H_i. Let $\mathscr{T} = \mathscr{T}_0$ be a system of H_0 and for $i = 1, 2, \cdots, n$ let $\mathscr{T}_i$ be a system of H_i reducing into $\mathscr{T}_{i-1}$. Thus, $N(\mathscr{T}_n)$ is a subgroup of H and $\mathscr{T}_n$ is a system of G reducing into H.

Let $x_1, x_2, \cdots, x_s$ be coset representatives of $N(\mathscr{T}_n)$ in H, where $s = |H : N(\mathscr{T}_n)|$. Therefore, the s systems $\mathscr{T}_n^{x_i}$, $i = 1, \cdots, s$, are s distinct systems of G which reduce into H. Furthermore, by their construction these systems are conjugate by elements of H. Thus, if we can show that there are exactly s systems of G reducing into H then we are done.

However, the number of systems of G reducing into H is equal to

$$\frac{Z_0(H)\pi(G)}{|G:H|} = \frac{1 \cdot |G:N(\mathscr{T}_n)|}{|G:H|} = |H:N(\mathscr{T}_n)| = s,$$

where the invariants $Z_0(H)$, $\pi(G)$ are those described by Carter [4], p. 541.

The next step in the proof of the theorem is the

LEMMA. *Let D be a system normalizer and C be a Carter subgroup of the solvable group G. If D is a subgroup of C then there is a system of G normalized by D and reducing into C.*

Proof: In order to accomplish the proof we need only show that when p is a prime there is a p-complement S of G which is normalized by D and such that $S \cap C$ is a p-complement of C. However, C is a nilpotent group so that if C^p is the p-complement of C and D^p is the p-complement of D then D^p is contained in C^p. Thus, the lemma is a direct consequence of Theorem B.

We can now complete the proof of Theorem 2. Let $\mathscr{S}$ and $\mathscr{S}_1$ be systems of G with normalizers D and D_1 respectively such that $\mathscr{S}$ and $\mathscr{S}_1$ reduce into C. This can be done by the previous lemma. By the other lemma there is $y \in C$ such that $\mathscr{S}^y = \mathscr{S}_1$. Hence, $D_1 = N(\mathscr{S}_1) = N(\mathscr{S})^y = D^y$ and the theorem is proved.

We now may go on with the proof of Theorem 3. The second part of this theorem follows immediately from the previous theorem. Next, let C and C^x, for $x \in G$, contain D. Hence, D and $D^{x^{-1}}$ are system normalizers of G contained in C. By Theorem 2 there exists $y \in C$ such that $D^y = D^{x^{-1}}$. Hence, $D^{yx} = D$ so $yx \in N(D)$. Thus $C^x = C^{yx}$ and each Carter subgroup containing D is obtained by conjugating C by an element of $N(D)$. The number of such subgroups is $|N(D):N(D) \cap N(C)|$. However, $N(C) = C$ so this is in fact just $|N(D):N_C(D)|$ as was to be shown.

As for the last part of the theorem, let $\mathscr{S}$ be a system of G normalized by D and reducing into C. The systems of G reducing into C are, as we have shown above, $\mathscr{S}^{x_i}$ where $x_1, \cdots, x_s$ is a set of coset representatives of D in C. But $N(\mathscr{S}^{x_i}) = N(\mathscr{S})^{x_i}$ and this equals D if and only if $D^{x_i} = D$ or if and only if $x_i \in N_C(D)$. This concludes the proof.

We now turn to Theorem 5 which shall be proved by induction on the group order. Let M be a minimal normal subgroup of G so that DM/M is a system normalizer of G/M and a subnormal subgroup of HM/M. Since CM/M is a Carter subgroup of G/M containing DM/M it follows that HM, and therefore H, is a subgroup of CM, by an application of the induction hypothesis.

Let $D = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$ be a subnormal series joining D and H. We again proceed by induction and show that each H_i is a subgroup of C. We may assume that H_i is a subgroup of C and shall prove that H_{i+1} is also based on the knowledge that H_{i+1} is a subgroup of MC.

Let C_p and C^p be the Sylow p-subgroup and p-complement of C where p is the prime such that M is a p-subgroup. The centralizer $E = C_M(C^p)$ is normalized by C_p since C_p normalizes C^p. Thus C_pE is a p-subgroup of G centralizing C^p and CE is a nilpotent subgroup of G containing C. Hence $CE = C$ and $E = C \cap M$.

Furthermore, if F if the Fitting subgroup of G then F centralizes M. The group G/F has nilpotent length at most two so that the system normalizers of G/F are Carter subgroups of G/F [2]. Thus, $DF = CF$ and D and C induce the same group of automorphisms on M.

Suppose that $x \in H_{i+1}$, $x \notin H_i$ and express $x = mc$ where $m \in M$ and $c \in C$. We shall show that $m \in M \cap C$ which will conclude the proof. In any case, if $h \in H_i$ then $(h, x) = h^{-1}x^{-1}hx \in H_i$ since x normalizes H_i. Thus $(h, x) \in C$. But

$$(h, x) = (h, mc) = (h, c)(h, m)^c \in C$$

so $(h, m) \in C$. Therefore, the coset of $C \cap M$ in M containing m is left fixed by H_i. Since D is contained in H_i which is in turn a subgroup of C, it follows that H_i and C induce the same group of automorphisms of M. Therefore, the coset of $C \cap M$ in M containing m is left fixed by C^p. Let $M_1/C \cap M$ be the subgroup of $M/C \cap M$ centralized by C^p. Hence, the group of automorphisms induced by C^p on M_1 in turn induce only the identity automorphism of $M_1/C \cap M$ and $C \cap M$. However, the group of all such automorphisms is a p-group and C^p is a p'-subgroup of G. Thus $M_1 = C \cap M$ and $m \in C \cap M$.

As for Theorem 6, we shall count in two ways the number of pairs $(\mathcal{S}, C)$ where $\mathcal{S}$ is a system of G, C is a Carter subgroup of G and $\mathcal{S}$ reduces into C. We remark that if $\mathcal{S}$ is reducible into C then $D = N(\mathcal{S})$ is a subgroup of C (see [2], p. 634). Therefore, the number of such pairs is the product of the number of system normalizers, the number of Carter subgroups containing a fixed system normalizer and the number of systems normalized by D and reducing into a fixed Carter subgroup containing D. This follows from the conjugacy of system normalizers and from the conjugacy under $N(D)$ of the Carter subgroups containing a system normalizer D. This product is

$$| G : N(D) | \, | N(D) : N_C(D) | \, | N_C(D) : D | = | G : D |$$

by Theorem 3. However, this must also be equal, as all systems are conjugate, to the product of the numbers of systems and the number of Carter subgroups into which each system is reducible. But the number of systems is $| G : D |$ so the theorem is proved.

Theorem 7 is another consequence of Theorem B. The subgroup D is contained in the hypercenter of $N(D)$ [5] and therefore normalizes every system of $N(D)$. Hence, if N^p is a p-complement of $N(D)$ then the Sylow p-subgroup D_p of D centralizes N^p and the p-complement D^p of D is a subgroup of N^p. We may choose a p-complement T^p of G which contains N^p and is normalized by D, by Theorem B. Let S^p be the p-complement of $\mathscr{S}$. Choose $x \in N(D)$ such that $(T^p)^x = S^p$. Hence,

$$S^p \cap N(D) = (T^p)^x \cap N(D) = (T^p \cap N(D))^x = (N^p)^x$$

is a p-complement of $N(D)$ and we are done.

Before beginning the proof of Theorem 8 we shall recall a few relevant facts. If H, K and L are subgroups of a group G and L is a normal subgroup of K then the projection of H into K/L is the subgroup of K/L consisting of those cosets of L in K which contain an element of H. With this definition, the product of the orders of the projections of H into the factors in a given chief series of G is equal to the order of H [7].

If $D = D_1 D_2 \cdots D_n$ is a subgroup then it has the covering property since any central chief factor of G is covered by one of the D_i. Therefore, the order of D is at least the product of the orders of the central factors in a chief series of G which is equal to the order of the system normalizers of G. On the other hand, D has order at most the product of the orders of $D_1, \cdots, D_n$ so D has order the order of a system normalizer. Consequently, by the remark in the previous paragraph, D avoids every eccentric chief factor of G. Furthermore, the term-by-term intersection of a chief series of G with D will be a central series for D so D is nilpotent and a direct product of its Sylow subgroups $D_1, \cdots, D_n$. Thus, granting Theorem 9, the proof is reduced to just another simple application of Theorem B, since each D_i is assumed to normalize some p_i-complement of G.

In the proof of Theorem 9, as in the above paragraph, we have that H is a nilpotent subgroup of G. If the Sylow p-subgroup H_p of H, for a prime p, normalizes a p-complement of G then, by Theorem B, H_p normalizes a p-complement of G containing the p-complement H^p of H. Hence, H normalizes a p-complement of G.

The last of these ten theorems is a corollary to Theorem 5. Let D be a system normalizer of G contained in M and normalizing a system of M. Thus D is a subgroup of a system normalizer E of M. But E is a subgroup of a Carter subgroup B of M so that B is a nilpotent subgroup containing D and thus containing it as a subnormal subgroup. Hence, B is a subgroup of C, the unique Carter subgroup containing D, by Theorem 5. Therefore, $C \cap M$ contains B and so contains it as a subnormal subgroup. Since B is of course self-normalizing in M we have $B = C \cap M$.

4. An Example

In order to construct the group needed for the proof of Theorem C we shall first analyze a very special type of situation. A more general analysis would be of interest in relation to a determination of conditions sufficient for extendibility of subgroups as discussed in the introduction.

If A is a group of automorphisms of a group C and H is a subgroup of G invariant under all elements of A then A permutes the cosets Hg of H in G. If $(Hg)^a = Hg$ for all $a \in A$ then Hg is called a fixed coset. In using this definition below we shall only be interested in the case where A is a group of inner automorphisms.

Before stating the special result we shall need we shall separately describe the assumptions.

Hypotheses 1. *P and E are p-subgroups and K is a p'-subgroup of the group G.*

2. *P and PK are normal subgroups of G.*
3. *E normalizes K and intersects PE trivially.*
4. *H is a subgroup of K normalized by E.*
5. *$C = C_P(H)$ and $D = C_P(K)$.*

Lemma. *With the above hypotheses, if there is a coset of C in P fixed by E which contains no elements of a coset of D in P fixed by E then there is a conjugate of H normalized by E which is a subgroup of no conjugate of K normalized by E.*

Therefore, since K is clearly a p-complement of G, we will have proved Theorem C once we exhibit a solvable group G satisfying the hypotheses and assumptions of the lemma.

Proof: Let Cx be a coset of C in P fixed by E which contains no element of a coset of D in P fixed by E. We shall prove that H^x is the desired conjugate of H.

As a first step, we shall show that H^x is in fact normalized by E. Indeed, if $e \in E$ then

$$(H^x)^e = (H^e)^{e^{-1}xe} = H^{e^{-1}xe} = H^{yx} = H^x$$

where $x^e = yx$ for $y \in C$ since $(Cx)^e = Cx$.

Suppose next, that H^x is in fact contained in a conjugate K^z of K which is normalized by E. We shall see now that we may assume that z lies in a coset of D in P fixed by E. However, K is normal in KE so

$$N(K) = KE \cdot N_P(K) = KED$$

and we may assume that $z \in P$. Furthermore, $K^z = (K^z)^e$ for all $e \in E$ and

$$K^z = K^{ze} = (K^e)^{e^{-1}ze} = K^{e^{-1}ze}.$$

Thus, $z^e z^{-1} \in N(K) \cap P = D$ so $z^e \in Dz$ and $(Dz)^e = Dz$.

Any coset of P in PK contains a unique element of K^z; in fact, this is true for any p-complement of PK. But H^x and H^z are subgroups of K^z and if $h \in H$ then h^x and h^z lie in the same coset of P in PK since x and z are elements of P. Thus, $h^x = h^z$, $H^x = H^z$, $Cx = Cz$ so $z \in Cx$, a contradiction.

We now proceed with the construction of the example. Form the wreath product of a quaternion subgroup of order eight and a cyclic group of order three. This group has a normal subgroup $P = Q_1 \times Q_2 \times Q_3$, the direct product of three quaternion groups, and a Sylow 3-subgroup L which permutes Q_1, Q_2 and Q_3 cyclically. The quaternion group of order eight has a group of automorphisms which is isomorphic to the symmetric group on three letters. Therefore, PL has a group A of six automorphisms which leave invariant each Q_i, inducing the automorphisms described, and leave L elementwise fixed. Let $G = PLA$ be the splitting extension of PL by A. If H is a Sylow 3-subgroup of A, E is a Sylow 2-subgroup of A and $K = LH$ then G, P, H, K and E fulfill all the requirements laid down.

In fact, $C = C_P(H) = Q_1' \times Q_2' \times Q_3'$ is the derived group of P so that C is a normal subgroup of G. The centralizer of E in P/C has order eight so that there are eight cosets of C in P fixed by E. Furthermore, $D = C_P(K)$ is of order two, a subgroup of C and normal in G. There are at most eight cosets of D in P fixed by E since $C_P(E)$ has order eight and E is cyclic; this bound is in fact achieved, as may be verified directly. However, E leaves fixed two cosets of D in C so that there are at most four cosets of C in P which contain elements of cosets of D in P fixed by E.

5. Concluding Remarks

Several problems suggest themselves almost immediately. First, a survey of the p'-subgroups normalized by a p-subgroup E, at least up to a description of the p'-subgroups maximal with this property, would be of use in proving results like Theorem 1. Second, a study of the embedding of system normalizers in the subgroups, analogous to the Carter subgroups, discovered by Gaschütz might well lead to more results on extendibility. Another possible direction for further investigation might be the study of Carter subgroups of proper subgroups; this seems to be a very difficult problem. A final possibility is the study of solvable groups of p-length one for all primes p using Theorem 1 and earlier results [1]. It would be an achievement to push the theory of such groups to the level reached in the study of A-groups.

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